

# Localized Solitons of a (2+1)-dimensional Nonlocal Nonlinear Schrödinger Equation

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## Abstract

An integrable (2+1)-dimensional nonlocal nonlinear Schrödinger equation is discussed. The  $N$ -soliton solution is given by Gram type determinant. It is found that the localized  $N$ -soliton solution has interesting interaction behavior which shows change of amplitude of localized pulses after collisions.

## 1 Introduction

The nonlinear Schrödinger (NLS) equation,

$$i\psi_t = \psi_{xx} + \alpha|\psi|^2\psi, \quad (1)$$

is the most important soliton equation which is a widely used model for investigating the evolution of pulses in optical fiber and of surface gravity waves with

narrow-banded spectra in fluid [1]. The study of vector and nonlocal analogues of the NLS equation has received considerable attention recently from both physical and mathematical points of view [1, 2, 3, 4, 5, 6].

In this Letter, we discuss a (2+1)-dimensional nonlocal nonlinear Schrödinger (2DNNLS) equation:

$$iu_t = u_{xx} + 2u \int_{-\infty}^{\infty} |u|^2 dy, \quad (2)$$

where  $u = u(x, y, t)$  is a complex function and  $x, y, t$  are real. The Gram type determinant solution is presented and localized soliton interactions are studied.

## 2 Determinant Solution

Using the dependent variable transformation

$$u(x, y, t) = \frac{g(x, y, t)}{f(x, t)}, \quad u^*(x, y, t) = \frac{g^*(x, y, t)}{f(x, t)},$$

where  $f$  is real and  $*$  is complex conjugate, we have bilinear equations [7]

$$(D_x^2 - iD_t)g \cdot f = 0, \quad (3)$$

$$(D_x^2 + iD_t)g^* \cdot f = 0, \quad (4)$$

$$D_x^2 f \cdot f = 2 \int_{-\infty}^{\infty} g g^* dy. \quad (5)$$

These bilinear equations have the following Gram determinant solution which is the  $N$ -soliton solution of the 2DNNLS equation:

$$f = \begin{vmatrix} \mathcal{A}_N & I_N \\ -I_N & \mathcal{B}_N \end{vmatrix},$$

$$g = \begin{vmatrix} \mathcal{A}_N & I_N & \mathbf{e}_N^T \\ -I_N & \mathcal{B}_N & \mathbf{0}^T \\ \mathbf{0} & -\mathbf{a}_N & 0 \end{vmatrix}, \quad g^* = - \begin{vmatrix} \mathcal{A}_N & I_N & \mathbf{0}^T \\ -I_N & \mathcal{B}_N & \mathbf{a}_N^{*T} \\ -\mathbf{e}_N^* & \mathbf{0} & 0 \end{vmatrix},$$

where

$$\mathcal{A}_N = \begin{pmatrix} \frac{e^{\xi_1 + \xi_1^*}}{p_1 + p_1^*} & \frac{e^{\xi_1 + \xi_2^*}}{p_1 + p_2^*} & \cdots & \frac{e^{\xi_1 + \xi_N^*}}{p_1 + p_N^*} \\ \frac{e^{\xi_2 + \xi_1^*}}{p_2 + p_1^*} & \frac{e^{\xi_2 + \xi_2^*}}{p_2 + p_2^*} & \cdots & \frac{e^{\xi_2 + \xi_N^*}}{p_2 + p_N^*} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{e^{\xi_N + \xi_1^*}}{p_N + p_1^*} & \frac{e^{\xi_N + \xi_2^*}}{p_N + p_2^*} & \cdots & \frac{e^{\xi_N + \xi_N^*}}{p_N + p_N^*} \end{pmatrix},$$

$$\mathcal{B}_N = \begin{pmatrix} \frac{\int_{-\infty}^{\infty} a_1^* a_1 dy}{p_1^* + p_1} & \frac{\int_{-\infty}^{\infty} a_1^* a_2 dy}{p_1^* + p_2} & \cdots & \frac{\int_{-\infty}^{\infty} a_1^* a_N dy}{p_1^* + p_N} \\ \frac{\int_{-\infty}^{\infty} a_2^* a_1 dy}{p_2^* + p_1} & \frac{\int_{-\infty}^{\infty} a_2^* a_2 dy}{p_2^* + p_2} & \cdots & \frac{\int_{-\infty}^{\infty} a_2^* a_N dy}{p_2^* + p_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\int_{-\infty}^{\infty} a_N^* a_1 dy}{p_N^* + p_1} & \frac{\int_{-\infty}^{\infty} a_N^* a_2 dy}{p_N^* + p_2} & \cdots & \frac{\int_{-\infty}^{\infty} a_N^* a_N dy}{p_N^* + p_N} \end{pmatrix},$$

and  $I_N$  is the  $N \times N$  identity matrix,  $\mathbf{a}^T$  is the transpose of  $\mathbf{a}$ ,

$$\mathbf{a}_N = (a_1, a_2, \dots, a_N), \quad \mathbf{e}_N = (e^{\xi_1}, e^{\xi_2}, \dots, e^{\xi_N}), \quad \mathbf{0} = (0, 0, \dots, 0),$$

$$\xi_i = p_i x - i p_i^2 t, \quad \xi_i^* = p_i^* x + i p_i^{*2} t, \quad 1 \leq i \leq N,$$

and  $p_i$  is a complex wave number of  $i$ -th soliton and  $a_i \equiv a_i(y)$  is a complex phase function of  $i$ -th soliton.

Here, we show that eq.(5) has the above Gram determinant solution.

Let us denote the  $(i, j)$ -cofactor of the matrix

$$M = \begin{pmatrix} \mathcal{A}_N & I_N \\ -I_N & \mathcal{B}_N \end{pmatrix}$$

as  $\Delta_{ij}$ . Then the  $x$ -derivative of  $f = \det M$  is given by

$$\begin{aligned} f_x &= \sum_{i=1}^N \sum_{j=1}^N \Delta_{ij} \frac{\partial}{\partial x} \frac{e^{\xi_i + \xi_j^*}}{p_i + p_j^*} = \sum_{i=1}^N \sum_{j=1}^N \Delta_{ij} e^{\xi_i + \xi_j^*} \\ &= \begin{vmatrix} \mathcal{A}_N & I_N & \mathbf{e}_N^T \\ -I_N & \mathcal{B}_N & \mathbf{0}^T \\ -\mathbf{e}_N^* & \mathbf{0} & 0 \end{vmatrix}. \end{aligned} \tag{6}$$

In the Gram determinant expression of  $f$ , dividing  $i$ -th row by  $e^{\xi_i}$  and multiplying  $(N+i)$ -th column by  $e^{\xi_i}$  for  $i = 1, \dots, N$ , and dividing  $j$ -th column by  $e^{\xi_j^*}$  and multiplying  $(N+j)$ -th row by  $e^{\xi_j^*}$  for  $j = 1, \dots, N$ , we obtain another determinant expression of  $f$ ,

$$f = \det M',$$

where

$$M' = \begin{pmatrix} \mathcal{A}'_N & I_N \\ -I_N & \mathcal{B}'_N \end{pmatrix},$$

$$\mathcal{A}'_N = \begin{pmatrix} \frac{1}{p_1+p_1^*} & \cdots & \frac{1}{p_1+p_N^*} \\ \vdots & \ddots & \vdots \\ \frac{1}{p_N+p_1^*} & \cdots & \frac{1}{p_N+p_N^*} \end{pmatrix},$$

$$\mathcal{B}'_N = \begin{pmatrix} \frac{e^{\xi_1^*+\xi_1} \int_{-\infty}^{\infty} a_1^* a_1 dy}{p_1^*+p_1} & \cdots & \frac{e^{\xi_1^*+\xi_N} \int_{-\infty}^{\infty} a_1^* a_N dy}{p_1^*+p_N} \\ \vdots & \ddots & \vdots \\ \frac{e^{\xi_N^*+\xi_1} \int_{-\infty}^{\infty} a_N^* a_1 dy}{p_N^*+p_1} & \cdots & \frac{e^{\xi_N^*+\xi_N} \int_{-\infty}^{\infty} a_N^* a_N dy}{p_N^*+p_N} \end{pmatrix}.$$

Thus the  $x$ -derivative of  $f$  is also written as

$$\begin{aligned} f_x &= \sum_{i=1}^N \sum_{j=1}^N \Delta'_{N+i,N+j} \frac{\partial}{\partial x} \frac{e^{\xi_i^*+\xi_j} \int_{-\infty}^{\infty} a_i^* a_j dy}{p_i^*+p_j} = \sum_{i=1}^N \sum_{j=1}^N \Delta'_{N+i,N+j} e^{\xi_i^*+\xi_j} \int_{-\infty}^{\infty} a_i^* a_j dy \\ &= \int_{-\infty}^{\infty} \sum_{i=1}^N \sum_{j=1}^N \Delta'_{N+i,N+j} e^{\xi_i^*+\xi_j} a_i^* a_j dy \end{aligned}$$

where  $\Delta'_{ij}$  is the  $(i, j)$ -cofactor of  $M'$ . Therefore we have

$$\begin{aligned} f_x &= \int_{-\infty}^{\infty} \begin{vmatrix} \mathcal{A}'_N & I_N & \mathbf{0}^T \\ -I_N & \mathcal{B}'_N & \tilde{\mathbf{a}}_N^{*T} \\ \mathbf{0} & -\tilde{\mathbf{a}}_N & 0 \end{vmatrix} dy \\ &= \int_{-\infty}^{\infty} \begin{vmatrix} \mathcal{A}_N & I_N & \mathbf{0}^T \\ -I_N & \mathcal{B}_N & \mathbf{a}_N^{*T} \\ \mathbf{0} & -\mathbf{a}_N & 0 \end{vmatrix} dy, \end{aligned}$$

where

$$\tilde{\mathbf{a}}_N = (e^{\xi_1} a_1, \dots, e^{\xi_N} a_N).$$

By differentiating the above  $f_x$  by  $x$ , we get

$$f_{xx} = \int_{-\infty}^{\infty} \begin{vmatrix} \mathcal{A}_N & I_N & \mathbf{e}_N^T & \mathbf{0}^T \\ -I_N & \mathcal{B}_N & \mathbf{0}^T & \mathbf{a}_N^{*T} \\ -\mathbf{e}_N^* & \mathbf{0} & 0 & 0 \\ \mathbf{0} & -\mathbf{a}_N & 0 & 0 \end{vmatrix} dy.$$

On the other hand, using the Jacobi formula for determinant[7], we have

$$\begin{aligned}
gg^* &= \begin{vmatrix} \mathcal{A}_N & I_N \\ -I_N & \mathcal{B}_N \end{vmatrix} \times \begin{vmatrix} \mathcal{A}_N & I_N & \mathbf{e}_N^T & \mathbf{0}^T \\ -I_N & \mathcal{B}_N & \mathbf{0}^T & \mathbf{a}_N^{*T} \\ -\mathbf{e}_N^* & \mathbf{0} & 0 & 0 \\ \mathbf{0} & -\mathbf{a}_N & 0 & 0 \end{vmatrix} \\
&- \begin{vmatrix} \mathcal{A}_N & I_N & \mathbf{e}_N^T \\ -I_N & \mathcal{B}_N & \mathbf{0}^T \\ -\mathbf{e}_N^* & \mathbf{0} & 0 \end{vmatrix} \times \begin{vmatrix} \mathcal{A}_N & I_N & \mathbf{0}^T \\ -I_N & \mathcal{B}_N & \mathbf{a}_N^{*T} \\ \mathbf{0} & -\mathbf{a}_N & 0 \end{vmatrix}.
\end{aligned}$$

Here we note that the  $y$ -dependence in right-hand side appears only in the last row and last column of the second determinant in each term. Thus we obtain

$$\int_{-\infty}^{\infty} gg^* dy = ff_{xx} - f_x f_x,$$

which is a bilinear equation (5).

Since eqs.(3) and (4) are bilinear equations for the NLS equation and do not include  $y$ , we can prove in the same way in the NLS equation that the above Gram determinant solution satisfies the bilinear identities (3) and (4), i.e., we can show easily that the bilinear equations (3) and (4) are made from a pair of Jacobi identities, respectively.

### 3 Localized Solitons

Using the above formula, we can make 1-soliton solution as follows:

$$u = \frac{g}{f} = \frac{a_1 e^{\xi_1}}{1 + \frac{\int_{-\infty}^{\infty} a_1^* a_1 dy}{(p_1^* + p_1)^2} e^{\xi_1 + \xi_1^*}}, \quad u^* = \frac{g^*}{f} = \frac{a_1^* e^{\xi_1^*}}{1 + \frac{\int_{-\infty}^{\infty} a_1^* a_1 dy}{(p_1^* + p_1)^2} e^{\xi_1 + \xi_1^*}}, \quad (7)$$

where

$$f = \begin{vmatrix} \frac{e^{\xi_1 + \xi_1^*}}{p_1 + p_1^*} & 1 \\ -1 & \frac{\int_{-\infty}^{\infty} a_1^* a_1 dy}{p_1^* + p_1} \end{vmatrix} = 1 + \frac{\int_{-\infty}^{\infty} a_1^* a_1 dy}{(p_1^* + p_1)^2} e^{\xi_1 + \xi_1^*},$$

$$g = \begin{vmatrix} \frac{e^{\xi_1 + \xi_1^*}}{p_1 + p_1^*} & 1 & e^{\xi_1} \\ -1 & \frac{\int_{-\infty}^{\infty} a_1^* a_1 dy}{p_1^* + p_1} & 0 \\ 0 & -a_1 & 0 \end{vmatrix} = a_1 e^{\xi_1},$$

$$g^* = - \begin{vmatrix} \frac{e^{\xi_1 + \xi_1^*}}{p_1 + p_1^*} & 1 & 0 \\ -1 & \frac{\int_{-\infty}^{\infty} a_1^* a_1 dy}{p_1^* + p_1} & a_1^* \\ -e^{\xi_1^*} & 0 & 0 \end{vmatrix} = a_1^* e^{\xi_1^*}.$$

If we choose  $a_1(y) = \alpha_1 \text{sech}(k(y + \eta_0))$  where  $\alpha_1$  is a complex number and  $k$  and  $\eta_0$  are real numbers,

$$u = \frac{\alpha_1 \text{sech}(k(y + \eta_0)) e^{\xi_1}}{1 + \frac{(2/k)|\alpha_1|^2}{(p_1^* + p_1)^2} e^{\xi_1 + \xi_1^*}} = \frac{\alpha_1}{2\sqrt{A}} \text{sech}(ky + \eta_0) \text{sech}\left(\frac{\xi_1 + \xi_1^*}{2} + \frac{1}{2} \log A\right) e^{\frac{\xi_1 - \xi_1^*}{2}},$$

where  $A = \frac{(2/k)|\alpha_1|^2}{(p_1^* + p_1)^2}$ . In this case, we have a localized pulse as shown in figure 1.

If we choose  $a_1(y) = \alpha_1 \text{sech}(k(y + \eta_1)) + \alpha_2 \text{sech}(k(y + \eta_2))$  where  $\alpha_1$  and  $\alpha_2$  are complex numbers and  $k$ ,  $\eta_1$  and  $\eta_2$  are real numbers,

$$u = \frac{(\alpha_1 \text{sech}(k(y + \eta_1)) + \alpha_2 \text{sech}(k(y + \eta_2))) e^{\xi_1}}{1 + \frac{(2/k)(|\alpha_1|^2 + |\alpha_2|^2) + 4(\eta_1 - \eta_2)(\alpha_1 \alpha_2^* + \alpha_1^* \alpha_2) / (e^{k(\eta_1 - \eta_2)} - e^{-k(\eta_1 - \eta_2)})}{(p_1^* + p_1)^2} e^{\xi_1 + \xi_1^*}}$$

$$= \frac{1}{2\sqrt{A}} (\alpha_1 \text{sech}(k(y + \eta_1)) + \alpha_2 \text{sech}(k(y + \eta_2)))$$

$$\times \text{sech}\left(\frac{\xi_1 + \xi_1^*}{2} + \frac{1}{2} \log A\right) e^{\frac{\xi_1 - \xi_1^*}{2}},$$

where  $A = \frac{(2/k)(|\alpha_1|^2 + |\alpha_2|^2) + 4(\eta_1 - \eta_2)(\alpha_1 \alpha_2^* + \alpha_1^* \alpha_2) / (e^{k(\eta_1 - \eta_2)} - e^{-k(\eta_1 - \eta_2)})}{(p_1^* + p_1)^2}$ . We see two localized pulses in figure 2. These two localized pulses travel parallel to the  $x$ -axis. With  $a_1(y) = \sum_j^M \alpha_j \text{sech}(k(y + \eta_j))$ , we can see  $M$ -localized pulses travelling parallel to the  $x$ -axis. We call this  $M$ -localized pulse the  $(1, M)$ -localized pulse solution. In the general case of pulse solutions generated from the  $N$ -soliton formula, it is named by  $(N, M)$ -localized pulse solution.

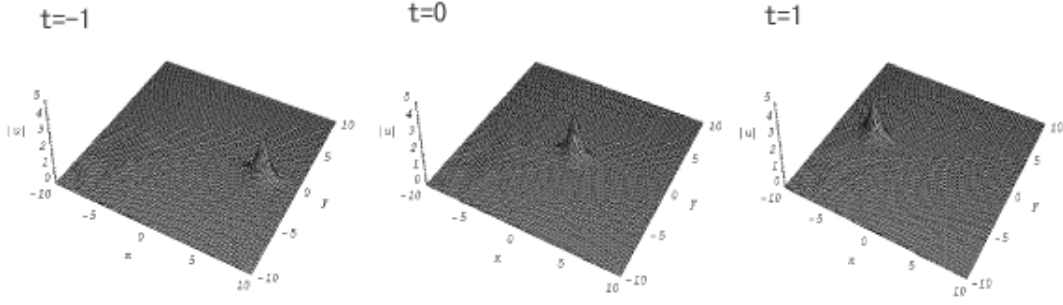


Figure 1: 1-soliton solution.  $\alpha_1 = 1 + 2i, p_1 = 2 + 3i, k = 3, \eta_0 = 0$ .

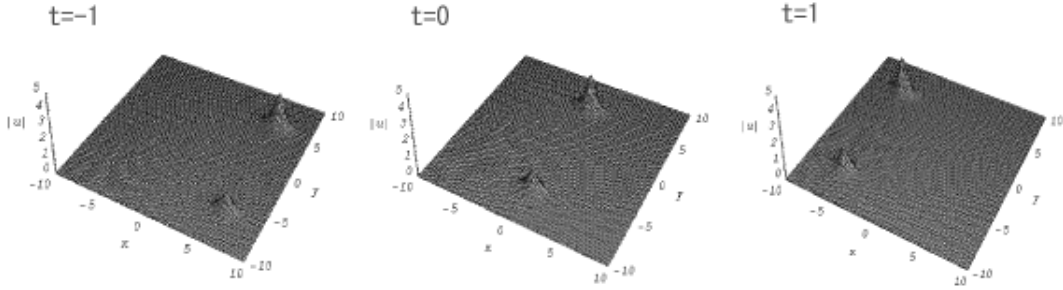


Figure 2: 1-soliton solution.  $\alpha_1 = 1 + 2i, \alpha_2 = 1/2 + i, p_1 = 2 + 3i, k = 3, \eta_1 = -6, \eta_2 = 6$ .

Next, we consider the case of  $N = 2$ , i.e. 2-soliton solution. Using the determinant form of  $N$ -soliton solution, we have

$$\begin{aligned}
 f &= \begin{vmatrix} \frac{e^{\xi_1 + \xi_1^*}}{p_1 + p_1^*} & \frac{e^{\xi_1 + \xi_2^*}}{p_1 + p_2^*} & 1 & 0 \\ \frac{e^{\xi_2 + \xi_1^*}}{p_2 + p_1^*} & \frac{e^{\xi_2 + \xi_2^*}}{p_2 + p_2^*} & 0 & 1 \\ -1 & 0 & \frac{\int_{-\infty}^{\infty} a_1^* a_1 dy}{p_1^* + p_1} & \frac{\int_{-\infty}^{\infty} a_1^* a_2 dy}{p_1^* + p_2} \\ 0 & -1 & \frac{\int_{-\infty}^{\infty} a_2^* a_1 dy}{p_2^* + p_1} & \frac{\int_{-\infty}^{\infty} a_2^* a_2 dy}{p_2^* + p_2} \end{vmatrix} \\
 &= 1 + \frac{c_{11}}{p_1^* + p_1} e^{\xi_1 + \xi_1^*} + \frac{c_{12}}{p_1^* + p_2} e^{\xi_1 + \xi_2^*} + \frac{c_{21}}{p_2^* + p_1} e^{\xi_2 + \xi_1^*} + \frac{c_{22}}{p_2^* + p_2} e^{\xi_2 + \xi_2^*} \\
 &\quad + \left( \frac{c_{12}c_{21} - c_{11}c_{22}}{(p_2^* + p_1)(p_1^* + p_2)} + \frac{c_{11}c_{22} - c_{12}c_{21}}{(p_1^* + p_1)(p_2^* + p_2)} \right) e^{\xi_1 + \xi_2 + \xi_1^* + \xi_2^*},
 \end{aligned}$$

$$\begin{aligned}
g &= \begin{vmatrix} \frac{e^{\xi_1+\xi_1^*}}{p_1+p_1^*} & \frac{e^{\xi_1+\xi_2^*}}{p_1+p_2^*} & 1 & 0 & e^{\xi_1} \\ \frac{e^{\xi_2+\xi_1^*}}{p_2+p_1^*} & \frac{e^{\xi_2+\xi_2^*}}{p_2+p_2^*} & 0 & 1 & e^{\xi_2} \\ -1 & 0 & \frac{\int_{-\infty}^{\infty} a_1^* a_1 dy}{p_1^*+p_1} & \frac{\int_{-\infty}^{\infty} a_1^* a_2 dy}{p_1^*+p_2} & 0 \\ 0 & -1 & \frac{\int_{-\infty}^{\infty} a_2^* a_1 dy}{p_2^*+p_1} & \frac{\int_{-\infty}^{\infty} a_2^* a_2 dy}{p_2^*+p_2} & 0 \\ 0 & 0 & -a_1 & -a_2 & 0 \end{vmatrix} \\
&= a_1 e^{\xi_1} + a_2 e^{\xi_2} + \frac{(c_{12}a_1 - c_{11}a_2)(p_1 - p_2)}{(p_1^* + p_1)(p_1^* + p_2)} e^{\xi_1+\xi_1^*+\xi_2} \\
&\quad + \frac{(c_{22}a_1 - c_{21}a_2)(p_1 - p_2)}{(p_2^* + p_1)(p_2^* + p_2)} e^{\xi_2+\xi_2^*+\xi_1}, \\
g^* &= - \begin{vmatrix} \frac{e^{\xi_1+\xi_1^*}}{p_1+p_1^*} & \frac{e^{\xi_1+\xi_2^*}}{p_1+p_2^*} & 1 & 0 & 0 \\ \frac{e^{\xi_2+\xi_1^*}}{p_2+p_1^*} & \frac{e^{\xi_2+\xi_2^*}}{p_2+p_2^*} & 0 & 1 & 0 \\ -1 & 0 & \frac{\int_{-\infty}^{\infty} a_1^* a_1 dy}{p_1^*+p_1} & \frac{\int_{-\infty}^{\infty} a_1^* a_2 dy}{p_1^*+p_2} & a_1^* \\ 0 & -1 & \frac{\int_{-\infty}^{\infty} a_2^* a_1 dy}{p_2^*+p_1} & \frac{\int_{-\infty}^{\infty} a_2^* a_2 dy}{p_2^*+p_2} & a_2^* \\ -e^{\xi_1^*} & -e^{\xi_2^*} & 0 & 0 & 0 \end{vmatrix} \\
&= a_1^* e^{\xi_1^*} + a_2^* e^{\xi_2^*} + \frac{(c_{21}a_1^* - c_{11}a_2^*)(p_1^* - p_2^*)}{(p_1^* + p_1)(p_1^* + p_2^*)} e^{\xi_1+\xi_1^*+\xi_2^*} \\
&\quad + \frac{(c_{22}a_1^* - c_{12}a_2^*)(p_1^* - p_2^*)}{(p_2^* + p_1^*)(p_2^* + p_2)} e^{\xi_2+\xi_2^*+\xi_1^*},
\end{aligned}$$

where  $c_{ij} = \int_{-\infty}^{\infty} a_i^* a_j dy / (p_i^* + p_j)$ .

To make four localized pulses, i.e. (2, 2)-localized pulse solution, we consider  $a_i(y) = \sum_{j=1}^2 \alpha_{2(i-1)+j} \text{sech}(k(y + \eta_j))$ . Then  $c_{ij}$  is given as follows.

$$\begin{aligned}
c_{ij} &= \frac{(2/k)(\alpha_{2(i-1)+1}^* \alpha_{2(j-1)+1} + \alpha_{2(i-1)+2}^* \alpha_{2(j-1)+2})}{(p_i^* + p_j)} \\
&\quad + \frac{4(\eta_1 - \eta_2)(\alpha_{2(j-1)+1} \alpha_{2(i-1)+2}^* + \alpha_{2(i-1)+1}^* \alpha_{2(j-1)+2})}{(e^{k(\eta_1 - \eta_2)} - e^{-k(\eta_1 - \eta_2)})(p_i^* + p_j)}.
\end{aligned}$$

Figure 3 is an example of (2, 2)-localized pulse solution. It is observed that 4 localized pulses suddenly change the height of pulses after a collision. Each pair



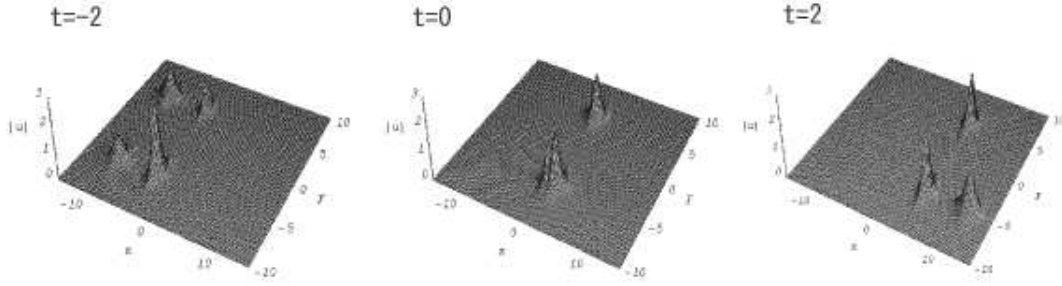


Figure 3: (2,2)-localized pulse solution.  $\alpha_1 = 1 + i, \alpha_2 = 1, \alpha_3 = 1, \alpha_4 = 1, p_1 = 3/2 - 5i/2, p_2 = 3 - i, k = 2, \eta_1 = -5, \eta_2 = 5$ .

of pulses on lines parallel to the  $x$ -axis collides, then the total mass of pulses is redistributed. In the case of figure 3, the height of a localized pulse become very small after a collision. Although there is a distance between two pulses on a line parallel to the  $x$ -axis and other two pulses on another line, the collision causes an effect of 4-pulse interaction. As this example, solutions of the 2DNNLS equation have very complicated and interesting properties.

## 4 Conclusion

We have discussed an integrable 2DNNLS equation and shown that the  $N$ -soliton solution of the 2DNNLS equation is given by the Gram type determinant and solutions can be localized in  $x$ - $y$  plane.

Note that the integrable 2DNNLS equation discussed in this Letter can be considered as the vector NLS equation with infinitely many components [8, 9, 10, 1]. This fact suggests that the vector soliton equations can produce nonlocal multi-dimensional soliton equations having localized pulses.

It should be noted that a model for second harmonic generation, i.e., quadratic solitons, was discussed in the paper by Nikolov et al., and they discussed the relationship between a nonlocal soliton equation and a vector soliton system [5]. Finding physical systems which could be described by the 2DNNLS equation is an interesting problem.

**Note added in proof:** After the acceptance of this Letter for publication, the

authors noticed the 2DNNLS equation (2) is equivalent to eq.(7.86) in ref.[11]. However, as far as we know, the N-soliton solution has not been obtained so far. The authors thank Dr. Takayuki Tsuchida for letting us know the paper by Zakharov [11].

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